A Hierarchy in Partial Differential Equations on Material Modeling

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Abstract

Various elasticity theories including linear and nonlinear theories on material modeling are reviewed. In the case of linear theories, a structure of hierarchy in the governing partial differential equations (PDEs) are observed. The structure of PDE hierarchy includes two sets of comparisons: (1) homogeneous materials versus nonhomogeneous materials, and (2) classical linear elasticity theory versus strain-gradient elasticity theory. We then found that crack problems can be used to simplify the formidable look of the governing PDEs. We also show that the fourth order PDE in the higher order strain-gradient elasticity theory converges to the second order PDE in classical linear elastic fracture mechanics (CLEFM). In the case of nonlinear theories, we observe that some nonlinear elasticity theory may not be applicable to formulate crack problems.

Keywords: Functionally graded material, Strain-gradient elasticity, Linear elastic fracture mechanics, Partial differential equation

1. Introduction

There are three the most basic field equations in linear elasticity:

\[ \epsilon = \frac{1}{2}(\nabla u + \nabla u^T), \]  
\[ \sigma = 2\mu \varepsilon + \lambda (\text{tr} \varepsilon) I, \]  

and

\[ \nabla \cdot \sigma = 0. \]  

Strain-displacement relations in equation (1) are under the assumption of small (infinitesimal) deformation, where \( u \) is the displacement vector, and \( \varepsilon \) is the strain tensor. Elastic constitutive law (Hooke’s law) in (2) is based upon the linear and local assumption between the stress tensor \( \sigma \) and strain tensor \( \varepsilon \), where \( \lambda \) is the Lamé’s constant, and \( \mu \) is the shear modulus. Under the condition of absence of body force the equilibrium equation (3) is a physical law stating the force must be balanced. Combining the above three field equations one obtains the governing system of PDE in linear isotropic elasticity, also known as Navier’s equations:

\[ \mu \nabla^2 u + (\lambda + \mu) \nabla (\nabla \cdot u) = 0. \]
In developing different elasticity theories, certainly one may relax the assumption(s) that are assumed in equations (1) and (2). For instance, without the assumption of small deformation, strain-displacement relations in (1) take the form
\[ \epsilon = \frac{1}{2}(\nabla u + \nabla u^T) + \nabla u \nabla u^T. \] (5)

Then, clearly, even under the linear relationship of the Hooke’s law in equation (2), the resulting “Navier’s equations” will be nonlinear.

As for another example, in the field of “functionally graded materials”, the Lamé moduli \( \lambda \) and \( \mu \) in equation (2), instead of being constant, are some functions of the material point \( x = (x, y, z) \), i.e. \( \lambda \equiv \lambda(x) \) and \( \mu \equiv \mu(x) \), then one should write equation (2) as
\[ \sigma = 2\mu(x) \epsilon + \lambda(x)(\text{tr}\epsilon)I. \] (6)

One may wonder if the corresponding governing PDEs still take the form as the Navier equations described in (4)? In addition to the question what the “changed Navier equation” to be, we also ask the following questions.
(i) Under the relaxation of the assumption, what governing PDEs do we have as compared to Navier equation (4)?
(ii) The governing PDEs may have formidable look; is there a way that we can simplify them?
(iii) Under the given boundary conditions, do the governing PDEs have solution(s)? That is, existence of solution(s)?
(iv) How to find the solution(s) to the corresponding boundary-value problem?
(iv) Will the solution be physical?

In this short paper we will not be able to answer all of the above questions. Even if we answer some of the questions, it will be brief, and the focus is on the three field equations and the governing PDEs. We address the FGMs in section 2; strain-gradient elasticity theory combined with FGMs is discussed in section 3; a table demonstrating the hierarchy structure of the governing PDEs and their corresponding fundamental solutions is concluded in section 4.

2. Functionally Graded Materials

For the sake of a more clear notation we write the constitutive equation (2) and the Navier’s equation (4), respectively, as \( \sigma = 2\mu_0 \epsilon + \lambda_0(\text{tr}\epsilon)I \) and \( \mu_0 \nabla^2 u + (\lambda_0 + \mu_0)\nabla(\nabla \cdot u) = 0 \), where the Lamé moduli \( \lambda_0 \) and \( \mu_0 \) are constant.

If the Lamé moduli are not constant, that is, \( \lambda \) and \( \mu \) in equation (2) are some functions of the material point \( x \), \( \lambda \equiv \lambda(x) \) and \( \mu \equiv \mu(x) \), and the constitutive relation is described by equation (6), then one can easily derive the Navier’s equation to be
\[ \mu(x)\nabla^2 u + [\lambda(x) + \mu(x)]\nabla(\nabla \cdot u) + (\nabla u + \nabla u^T) \nabla \mu(x) + (\nabla \cdot u) \nabla \lambda(x) = 0. \] (7)

Equation (7) can be considered as a perturbation of the Navier-Cauchy equations (4) for homogeneous materials. Comparing equations (7) and (4), one can observe that the perturbation brings in only the lower (first) order of differential operators, while the highest (second) order of differential operators have been preserved. If the Lamé moduli \( \lambda \) and \( \mu \) are constant, then equation (7) “returns back” to equation (4).

There has been a lot of work addressing the solutions to the PDE (7), particularly in FGM community. One popular function form for the Lamé moduli are of the exponential form \([3, 5]\)
\[ \mu(x) = \mu_0e^{\beta x} \text{ and } \lambda(x) = \lambda_0e^{\beta x}, \] (8)
in which \( \beta \) stands for the material gradation parameters.

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1 Simply replacing the Lamé moduli \( \lambda_0 \) and \( \mu_0 \) in equation (4) by \( \lambda(x) \) and \( \mu(x) \), i.e.,
\[ \mu(x) \nabla^2 u + [\lambda(x) + \mu(x)]\nabla(\nabla \cdot u) = 0, \] is NOT correct.
3. Strain-Gradient Elasticity

We briefly address the constitutive relations for strain-gradient elasticity in both homogeneous and functionally graded materials (FGMs) modeled as nonhomogeneous materials. To cope with the higher order tensors arise in the linear strain-gradient elasticity we use the index notation in this section.

For classical linear elasticity, the constitutive relations between the Cauchy stresses \( \sigma_{ij} \) and strains \( \varepsilon_{ij} \) have the same form for both homogeneous and nonhomogeneous materials. That is, \( \sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \), in which \( \delta_{ij} \) is the Kronecker-delta; the Lamé moduli \( \lambda \) and \( \mu \) can either be constant, \( \lambda = \lambda_0 \) and \( \mu = \mu_0 \), or they can be some functions of the material point \( \mathbf{x} \), \( \lambda \equiv \lambda(\mathbf{x}) \) and \( \mu \equiv \mu(\mathbf{x}) \). While the form of the constitutive relations is the same for homogeneous or graded materials in classical elasticity, such is not the case for strain-gradient elasticity where extra terms are generated due to the interaction of strain-gradient effect and material gradation. More specifically, for homogeneous materials, the constitutive relations in strain-gradient elasticity are (Exadaktylos et al. [4], Vardoulakis et al. [7]):

\[
\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} + 2\mu t'a_k \varepsilon_{ij} \,,
\]

where \( t' \) is a material characteristic length associated with surface energy gradient, \( \partial_k = \partial/\partial x_k \) is a differential operator, and \( \varepsilon_{i1} \), \( \varepsilon_{i2} \varepsilon_{i3} \) are two-dimensional, \( \varepsilon_{i1} \), \( \varepsilon_{i2} \varepsilon_{i3} \) have three displacement components, \( \mathbf{u} = (u, v, w) \), and the position variable \( \mathbf{x} \) is in three-dimensional space.

\[
\mu(\mathbf{x}) \nabla^2 (1 - \ell^2 \nabla^2) \mathbf{u} + (\lambda(\mathbf{x}) + \mu(\mathbf{x})) \nabla (1 - \ell^2 \nabla^2) \nabla \cdot \mathbf{u} + \nabla \mu(\mathbf{x}) \nabla \mathbf{u} + \nabla \lambda(\mathbf{x}) - \\
\ell^2 \left\{ \left( \frac{\partial}{\partial x} \mathbf{u} \right) \nabla \frac{\partial \mu(\mathbf{x})}{\partial x} + \left( \frac{\partial}{\partial y} \mathbf{u} \right) \nabla \frac{\partial \mu(\mathbf{x})}{\partial y} + \left( \frac{\partial}{\partial z} \mathbf{u} \right) \nabla \frac{\partial \mu(\mathbf{x})}{\partial z} - \nabla \mathbf{u} \nabla \lambda(\mathbf{x}) \cdot \nabla \mathbf{u} \right\} - \\
\ell^2 \left\{ \left( \frac{\partial}{\partial x} [(\nabla \cdot \mathbf{u}) \nabla \mu(\mathbf{x})] + \left( \frac{\partial}{\partial y} [(\nabla \cdot \mathbf{u}) \nabla \mu(\mathbf{x})] + \left( \frac{\partial}{\partial z} [(\nabla \cdot \mathbf{u}) \nabla \mu(\mathbf{x})] + (\nabla \cdot \mathbf{u}) \nabla \mu(\mathbf{x}) \right) = 0
\right.
\]

4. Concluding Remarks

The governing system of PDEs (11) has a formidable appearance, and certainly it poses a big challenge on obtaining a solution. However, by assuming that the Lamé moduli are of the exponential form described in equation (8), the gradation parameter \( \beta = (\beta, \gamma) \) and two-dimensional position variable \( \mathbf{x} \), the governing PDEs can be derived as shown in Table 1. We also obtain their corresponding fundamental solutions in Table 1, where \( r = ||\mathbf{x}|| = \sqrt{x^2 + y^2} \) and \( K_0 \) is the modified Bessel function.

The PDE hierarchy structure and the hierarchy structure of the fundamental solution can be clearly observed in Table 1. The most general form of PDE is derived to be

\[
\left( 1 - \beta \ell^2 \frac{\partial}{\partial x} - \gamma \ell^2 \frac{\partial}{\partial y} - \ell^2 \nabla^2 \right) \left( \nabla^2 + \beta \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y} \right) w = 0 ,
\]

and its fundamental solution can easily be obtained as

\[
e^{\beta(x-x') \gamma(y-y')} \frac{1}{2\pi} \left[ K_0 \left( \sqrt{\beta^2 + \gamma^2 + \frac{1}{\ell^2}} \right) r + K_0 \left( \sqrt{\beta^2 + \gamma^2 + \frac{1}{\ell^2}} \right) r \right] .
\]

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Table 1: PDE hierarchy and its fundamental solution.

<table>
<thead>
<tr>
<th>PDE</th>
<th>Fundamental solution</th>
</tr>
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<tbody>
<tr>
<td>( \nabla^2 w = 0 )</td>
<td>(-\frac{1}{2\pi} \log r )</td>
</tr>
<tr>
<td>( \left( \nabla^2 + \gamma \frac{\partial}{\partial y} \right) w = 0 )</td>
<td>( \frac{1}{2\pi} e^{\gamma(y-y')} K_0(</td>
</tr>
<tr>
<td>( \left( \nabla^2 + \beta \frac{\partial}{\partial y} \right) w = 0 )</td>
<td>( \frac{1}{2\pi} e^{\beta(z-z')} K_0(</td>
</tr>
<tr>
<td>( (1 - \ell^2 \nabla^2) \nabla^2 w = 0 )</td>
<td>(-\frac{1}{2\pi} \log r + \frac{1}{2\pi} K_0(r/\ell) )</td>
</tr>
<tr>
<td>( (1 - \gamma \ell^2 \frac{\partial}{\partial y} - \ell^2 \nabla^2)(\nabla^2 + \gamma \frac{\partial}{\partial y})w = 0 )</td>
<td>( \frac{e^{\gamma(y-y')}}{2\pi} \left[ K_0(</td>
</tr>
<tr>
<td>( (1 - \beta \ell^2 \frac{\partial}{\partial y} - \ell^2 \nabla^2)(\nabla^2 + \beta \frac{\partial}{\partial y})w = 0 )</td>
<td>( \frac{e^{\beta(z-z')}}{2\pi} \left[ K_0(</td>
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References